Birational geometry of Fano double covers

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We prove divisorial canonicity of Fano double hypersurfaces of general position.

Introduction

0.1. The main result. The symbol $\mathbb{P}$ stands for the projective space $\mathbb{P}^{M+1}$, $M \geq 6$. Fix a pair of integers $m \geq 3$, $l \geq 2$, satisfying the relation $m + l = M + 1$. Let

$$\mathcal{F} \subset \{(f,g) \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(m)) \times H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(2l))\}$$

be the Zariski open set of pairs of non-zero polynomials (f,g), for which the double cover $\sigma: V \to Q \subset \mathbb{P}$, branched over $W = W^* \cap Q$, is an irreducible variety, where $Q = \{f = 0\}$ is a hypersurface of dergee m, $W^* = \{g = 0\}$ is a hypersurface of degree $2l$. Set $\mathcal{F}_{\rm sm} \subset \mathcal{F}$ to be the open subset, corresponding to smooth double covers V. The aim of this paper is to prove the following fact.

Theorem 1. *There exists a non-empty Zariski open subset $\mathcal{F}_{\rm reg} \subset \mathcal{F}_{\rm sm}$ such that for any pair $(f,g) \in \mathcal{F}_{\rm reg}$ the corresponding variety V satisfies the property of divisorial canonicity: for any effective divisor $D \in |-nK_V|$, $n \geq 1$, the pair*

$$(V, \frac{1}{n}D)$$

has canonical singularities.

Recall that for $(f,g) \in \mathcal{F}_{\rm sm}$ the corresponding variety $V = V(f,g)$ is a primitive Fano variety of dimension $M \geq 6$, that is, $\mathrm{Pic} V = \mathbb{Z}K_V$. Canonicity of the pair $(V, \frac{1}{n}D)$ means that for any birational morphism $\varphi: V^+ \to V$ and any exceptional divisor $E \subset V^+$ the inequality

$$\nu_E(D) \leq na(E)$$

holds, where $a(E)$ is the discrepancy of E with respect to the model V, that is, the inequality, opposite to the Noether-Fano inequality. By linearity of this inequality one may always assume that D is a prime divisor, that is, irreducible and reduced.

0.2. Birational rigidity. The property of divisorial canonicity (the property (C)) was introduced in [1]. If the pair $(V, \frac{1}{n}D)$ is canonical for a general divisor $D \in \Sigma \subset |-nK_V|$ of any movable linear system Σ, then the variety V satisfies the

property of *movable* canonicity (the property (M)). Finally, if the pair $(V, \frac{1}{n}D)$ is log canonical for any divisor $D \in \Sigma \subset |-nK_V|$, then the variety V is *divisorially log canonical* (satisfies the property (L)). The movable canonicity is shown for many classes of Fano varieties, see [2] and the bibliography for that paper. The latter property is important because it immediately implies *birational rigidity* of the given variety.

Recall that a smooth projective rationally connected variety X is said to be *birationally superrigid*, if for any movable linear system Σ on X the equality

$$c_{\mathrm{virt}}(\Sigma) = c(\Sigma, X)$$

holds, where $c(\Sigma, X) = \sup\{t \in \mathbb{Q} | D + tK_X \in A^1_+ X, D \in \Sigma\}$ is the *threshold of canonical adjunction* (the symbol $A^1_+ X$ stands for the pseudoeffective cone of the variety X in $A^1_{\mathbb{R}} X = A^1 X \otimes \mathbb{R}$), whereas

$$c_{\mathrm{virt}}(\Sigma) = \inf_{X^+ \to X} \{c(\Sigma^+, X^+)\}$$

is the *virtual threshold of canonical adjunction*, the infimum is taken over all birational morphisms $X^+ \to X$ of smooth projective varieties, Σ^+ is the strict transform of Σ on X^+. If for any movable system Σ there is a birational self-map $\chi \in \mathrm{Bir}\, X$, providing the equality of the thresholds,

$$c_{\mathrm{virt}}(\Sigma) = c(\chi_*^{-1}\Sigma, X),$$

where $\chi_*^{-1}\Sigma$ is the strict transform of the system Σ with respect to χ, then the variety X is said to be *birationally rigid*. The property (M) and birational superrigidity of Fano double hypersurfaces were proven in [3].

The main geometric implication of birational rigidity for primitive Fano varieties is the absence of non-trivial structures of a rationally connected fiber space, that is, of rational dominant maps $\rho: X \dashrightarrow S$, the fiber of general position of which is rationally connected. For this reason, birationally rigid primitive Fano varieties are automatically non-rational. Besides, birational rigidity makes it possible to give an exhaustive description of birational maps of the given variety onto other varieties.

0.3. The theorem on direct products. The importance of divisorial canonicity is connected with the following theorem proven in [1] (the condition (C) implies the conditions (M) and (L) in an obvious way).

Theorem 2. *Assume that primitive Fano varieties $F_1, \ldots, F_K$, $K \geq 2$, satisfy the conditions (L) and (M). Then their direct product*

$$V = F_1 \times \ldots \times F_K$$

is a birationally superrigid variety.

Here are the main geometric consequences of birational rigidity of the direct product V (see [1, Corollary 1]).

(i) All structures of a rationally connected fiber space on the variety V are projections onto direct factors. More precisely, let $\beta\colon V^\sharp \to S^\sharp$ be a rationally connected fiber space and $\chi\colon V - - \to V^\sharp$ a birational map. Then there exist a set of indices

$$I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, K\}$$

and a birational map

$$\alpha\colon F_I = \prod_{i\in I} F_i - - \to S^\sharp$$

such that the following diagram commutes:

$$\begin{array}{ccccc} & V & \stackrel{\chi}{- - \to} & V^\sharp & \\ \pi_I & \downarrow & & \downarrow & \beta \\ & F_I & \stackrel{\alpha}{- - \to} & S^\sharp, & \end{array}$$

that is, $\beta \circ \chi = \alpha \circ \pi_I$, where $\pi_I\colon \prod\limits_{i=1}^{K} F_i \to \prod\limits_{i\in I} F_i$ is the natural projection onto a direct factor. In particular, on the variety V there are no structures of a fibration into rationally connected varieties of dimension strictly smaller than $\min\{\dim F_i\}$. In particular, V has no structures of a fibration into conics and rational surfaces.

(ii) The groups of birational and biregular self-maps of the variety V coincide:

$$\mathrm{Bir}\, V = \mathrm{Aut}\, V.$$

In particular, the group $\mathrm{Bir}\, V$ is finite.

(iii) The variety V is non-rational.

For a generic double cover $F \in \mathcal{F}_{\mathrm{reg}}$ we have (see [3]):

$$\mathrm{Bir}\, F = \mathrm{Aut} F = \mathbb{Z}/2\mathbb{Z}.$$

Therefore, for pair-wise non-isomorphic double covers of general position $F_1, \ldots, F_K \in \mathcal{F}_{\mathrm{reg}}$ we get

$$\mathrm{Bir}\, V = \mathrm{Aut}\, V = (\mathbb{Z}/2\mathbb{Z})^K.$$

On the other hand, for a double cover $F \in \mathcal{F}_{\mathrm{reg}}$ of general position the group $\mathrm{Bir}\, F^{\times K} = \mathrm{Aut} F^{\times K}$ is the extension

$$1 \to (\mathbb{Z}/2\mathbb{Z})^K \to \mathrm{Aut}\, F^{\times K} \to S_K \to 1,$$

where S_K is the symmetric group of permutations of K elements. This extension is in fact a semi-direct product: the group of automorphisms $\mathrm{Aut}\, V$ contains S_K as a subgroup permuting the factors of the direct product $F \times \ldots \times F$. The action of S_K on the subgroup $(\mathbb{Z}/2\mathbb{Z})^K$ is also obvious: the permutations $\mu \in S_K$ permute the generators $\sigma_1, \ldots, \sigma_K$ of the Galois groups of the factors F.

Another application of Theorem 1 is in that it allows to apply the linear method of proving birational rigidity [4] to investigating Fano fibrations $V/\mathbb{P}^1$ into double hypersurfaces [5,6], see §5 of this paper.

0.4. The structure of the paper. The open set $\mathcal{F}_{\rm reg}$ is defined by explicit conditions in §1. In §4 we show that this set is non-empty. Let $(f,g)\in\mathcal{F}_{\rm reg}$ be a fixed pair of polynomials, $V=V(f,g)$ the corresponding double cover. Assume that the claim of Theorem 1 does not hold for V, that is, for some effective divisor $D\in|-nK_V|$, $n\geq 1$, the pair $(V,\frac{1}{n}D)$ is not canonical. In the notations of Sec. 0.1, let $C=\varphi(E)\subset V$ be the centre of the singularity E on the variety V.

Proposition 0.1. *The following inequality holds:* $\operatorname{codim} C\geq 3$.

Proof. Assume the converse: $C\subset V$ is a subvariety of codimension 2. The Noether-Fano inequality implies immediately that $\operatorname{mult}_C D>n$. In particular, the divisor D does not coincide with the ramification divisor $R\subset V$ of the double cover σ (because R is a smooth divisor on V). Obviously, $R\cong W$, that is, R is a smooth complete intersection of codimension 2 in $\mathbb{P}$.

Restricting D onto R, we obtain an effective divisor Z on R, which is cut out by a hypersurface of degree n (in the sense of the identification $R\cong W$, mentioned above). Let Y be an irreducible component of the set $C\cap R$, $\operatorname{codim}_R Y=1$ (if $C\subset R$) or 2 (otherwise). We have the inequality

$$\operatorname{mult}_Y Z>n.$$

Now the cone method (see [7, Proposition 3.6]) gives a contradiction (the case when $\operatorname{codim}_R Y=1$ is excluded at once by the Lefschetz theorem for R), since $\dim Y=M-3\geq 3$. Q.E.D. for the proposition.

Let $o\in C$ be a point of general position, $\varphi\colon V^+\to V$ its blow up, $E=\varphi^{-1}(o)\subset V^+$ the exceptional divisor, D^+ the strict transform of the divisor D on V^+.

Proposition 0.2. *For some hyperplane $B\subset E$ the inequality*

$$\operatorname{mult}_o D+\operatorname{mult}_B D^+>2n. \qquad (1)$$

holds.

Proof is given in [1, §3].

Therefore, the proof of Theorem 1 will be complete if we show that for any point $o\in V$, where $V\in\mathcal{F}_{\rm reg}$ is a regular variety, any (prime) divisor $D\in|-nK_V|$ and any hyperplane $B\subset E$ the following inequality, which is opposite to (1), holds:

$$\operatorname{mult}_o D+\operatorname{mult}_B D^+\leq 2n. \qquad (2)$$

A proof of this inequality makes the heart of this paper (§§2-3).

0.5. Remarks. The concept of birational (super)rigidity is now universally accepted, but different authors use different definitions [2,8-10]. It seems that today the most precise definition is as follows: a projective rationally connected variety X is birationally rigid, if there exists a model $\widetilde{X}$, which is birational to X, satisfying the condition of Sec. 0.2, that is, for any movable system Σ on $\widetilde{X}$ there exists a birational self-map $\chi\in\operatorname{Bir}\widetilde{X}=\operatorname{Bir}X$, providing the equality of the thresholds $c_{\rm virt}(\Sigma)=c(\Sigma)$. (As the example of [10] shows, on the very variety X such a self-map χ may not exist.) In other words, the variety X is birationally rigid, if on some

its model $\widetilde{X}$ one can untwist all maximal singularities of movable linear systems by means of birational self-maps.

The present paper is based on the ideas and constructions developed in [1]. In [4] we started to investigate Fano double hypersurfaces, however the hardest case, when the point o lies on the ramification divisor of the morphism σ, was not considered there. For this reason in [4] an intermediate result was formulated on the canonicity of pairs $(V, \frac{1}{n}D)$, where V is a Fano double hypersurface.

Theorem 1 extends the action of the theorem on Fano direct products [1] to the double hypersurfaces. It can be interpreted as a statement on the global canonical threshold of the variety V: for a generic double hypersurface V the threshold is equal to 1. The global log canonical thresholds are of importance in differential geometry (existence of the Kähler-Einstein metric, see [11-13]).

The claim of Theorem 1 for $m = 1$ (the double spaces) was shown in [1], for $m = 2$ (the double quadrics) in [4].

Finally, combining the arguments of the present paper with the constructions of [14] in the spirit of the paper [1], one can prove in the same way the divisorial canonicity of iterated Fano double covers

$$V = V^{(l)} \to V^{(l-1)} \to \ldots \to V^{(1)} \to Q \subset \mathbb{P},$$

where $Q \subset \mathbb{P}^{M+k}$ is a Fano complete intersection of index ≥ 3, and each arrow is a double cover with a smooth branch divisor, and moreover $V = V^{(l)}$ is a primitive Fano variety.

1 Regular double covers

In this section we formulate the regularity conditions, defining the set $\mathcal{F}_{\rm reg}$. In §4 it is shown that it is non-empty, that is, that the regularity conditions make sense. The regularity conditions are local and differ essentially in the cases when the given point $o \in V$ lies on the ramification divisor of the morphism σ and when it does not.

1.1. The regularity conditions outside the branch divisor. We work in a fixed coordinate system $z_1, \ldots, z_{M+1}$ on $\mathbb{P}$ with the origin at the point $p = \sigma(o)$. The equation of the hypersurface Q is $f = q_1 + \ldots + q_m$, the equation of the hypersurface W^*, which cuts out on Q the branch divisor W, is $g = 1 + w_1 + \ldots + w_{2l}$. Following [3,14], let us consider the formal series

$$(1+t)^{1/2} = 1 + \sum_{i=1}^{\infty} \gamma_i t^i = 1 + \frac{1}{2}t - \frac{1}{8}t^2 + \ldots,$$

where

$$\gamma_i = (-1)^{i-1}\frac{(2i-3)!!}{2^i i!} = (-1)^{i-1}\frac{(2i-3)!}{2^{2i-2}i!(i-2)!},$$

and construct the corresponding formal series in the variables z_*:

$$\sqrt{g}=(1+w_1+\ldots+w_{2l})^{1/2}=1+\sum_{i=1}^{\infty}\gamma_i(w_1+\ldots+w_{2l})^i=$$

$$=1+\sum_{i=1}^{\infty}\Phi_i(w_1,\ldots,w_{2l}),$$

where $\Phi_i(w_1(z_*),\ldots,w_{2l}(z_*))$ are homogeneous polynomials of degree i in the variables z_*. Obviously,

$$\Phi_i(w_*)=\frac{1}{2}w_i+(\text{polynomial in }\ w_1,\ldots,w_{i-1}).$$

For instance, $\Phi_1(w_*)=\frac{1}{2}w_1$. Furthermore, for $j\geq 1$ set

$$[\sqrt{g}]_j=1+\sum_{i=1}^{j}\Phi_i(w_*(z_*))$$

and

$$g^{(j)}=g-[\sqrt{g}]_j^2.$$

It is easy to see that the first non-zero homogeneous component of the polynomial $g^{(j)}$ is of degree $j+1$. Denote it by the symbol h_{j+1}. Obviously,

$$h_{j+1}[g]=w_{j+1}+A_j(w_1,\ldots,w_j), \tag{3}$$

where the precise form of the polynomial A_j is of no interest for us. We say that the variety V is *regular* at the point o, if the following three regularity conditions (R1.1)-(R1.3) hold.

(R1.1) The sequence of homogeneous polynomials

$$q_1,\ldots,q_m,h_{l+1},\ldots h_{2l-1}$$

is regular in the local ring $\mathcal{O}_{0,\mathbb{C}^{M+1}}$, that is, the system of equations

$$q_1=\ldots=q_m=h_{l+1}=\ldots=h_{2l-1}=0$$

defines a one-dimensional set in $\mathbb{C}^{M+1}$, a finite set of lines, passing through the origin.

(R1.2) The linear span of any irreducible component of the closed algebraic set

$$q_1=q_2=q_3=0$$

in $\mathbb{C}^{M+1}$ is the hyperplane $q_1=0$.

(R1.3) Here we need to separate the two cases: $m=3$ and $m\geq 4$. For $m=3$ we require that the σ-preimage of a section of any irreducible component of the closed algebraic set $\overline{\{q_1=q_2=q_3=0\}}\subset Q$ by any anticanonical divisor $\sigma^{-1}(P)\ni o$,

containing the point o, should be irreducible (that is, the corresponding double cover should not break). Assume now that $m \geq 4$. In that case the regularity condition requires that the closed algebraic set

$$\sigma^{-1}(\overline{\{q_1 = q_2 = 0\}} \cap Q) \subset V$$

should be irreducible and any section of that set by an anticanonical divisor $\sigma^{-1}(P) \ni o$, where $P \ni p$ is some hyperplane,

- either is also irreducible and reduced,
- or breaks into two irreducible components $B_1 + B_2$, where $B_i = \sigma^{-1}(Q \cap S_i)$ is the σ-preimage of a section of Q by a plane $S_i \subset \mathbb{P}$ of codimension three and, moreover, $\mathrm{mult}_o B_i = \mathrm{mult}_p Q \cap S_i = 3$,
- or non-reduced and is of the form $2B$, where $B = \sigma^{-1}(Q \cap S)$ is the σ-preimage of a section of Q by a plane $S \subset \mathbb{P}$ of codimension three and, moreover, $\mathrm{mult}_o B = \mathrm{mult}_p(Q \cap S) = 3$.

Remark 1.1. As we will see in the proof of Proposition 1.1 below, the second and third options in the condition (R1.3) realize for a variety V of general position only when the quadric

$$q_2 \,|\, _{\{q_1=0\}\cap P}$$

is respectively, a pair of planes $S_1 \cup S_2$ or a double plane $2S$. For this reason for $M \geq 7$ the second and third options can be excluded.

1.2. The regularity conditions on the branch divisor. Assume that the point $o \in V$ lies on the ramification divisor of the morphism σ, $p = \sigma(o) \in W$. Let $\varphi_V \colon V^+ \to V$ and $\varphi_Q \colon Q^+ \to Q$ be the blow ups of the points o and p, respectively, $E_V = \varphi_V^{-1}(o) \subset V^+$ and $E_Q = \varphi_Q^{-1}(p) \subset Q^+$ the exceptional divisors. Let $W^+ \subset Q^+$ be the strict transform of the hypersurface W, $E_W = W^+ \cap E_Q$ a hyperplane in $E_Q \cong \mathbb{P}^{M-1}$. The symbols E_V^*, E_Q^* and E_W^* stand for the dual projective spaces. The natural embedding $\sigma^* \colon T_p^* W \hookrightarrow T_o^* V$ defines the embedding $\sigma^* \colon E_W^* \hookrightarrow E_V^*$. The rational map $\sigma^+ \colon V^+ \dashrightarrow Q^+$, induced by the morphism σ, defines the linear projection $\sigma_E^+ \colon E_V \dashrightarrow E_W \cong \mathbb{P}^{M-2}$, which is dual to that embedding.

In the coordinate form, let $z_1, \ldots, z_{M+1}$ be affine coordinates on $\mathbb{P}$ with the origin at the point p. The hypersurface Q is given by the equation

$$f(z_1, \ldots, z_{M+1}) = q_1 + \ldots + q_m,$$

where we can assume that $q_1 \equiv z_{M+1}$, so that the functions $z_1 \,|\, _Q, \ldots, z_M \,|\, _Q$ define on Q a system of local coordinates. The hypersurface W^* is given by the equation

$$g(z_1, \ldots, z_{M+1}) = w_1 + \ldots + w_{2l},$$

where we may assume that $w_1 = z_1$, so that the functions $z_2 \,|\, _W, \ldots, z_M \,|\, _W$ define on the branch divisor W a system of local coordinates. The variety V is given in $\mathbb{A}^{M+2}_{(y, z_1, \ldots, z_{M+1})}$ by the system of affine equations

$$y^2 - g(z_*) = f(z_*) = 0,$$

which is locally of the form

$$y^2 - z_1 + \ldots = z_{M+1} + \ldots = 0,$$

where the dots stand for the terms of order 2 and higher in z_*, so that the functions $y\,|\,_V, z_2\,|\,_V, \ldots, z_M\,|\,_V$ make a system of local coordinates on V with the origin at the point o. In these coordinates the linear projection σ_E^+ takes the form

$$(y, z_2, \ldots, z_M) \mapsto (z_2, \ldots, z_M).$$

Let $B \subset E_V$ be a hyperplane. There are two possible cases:

(A) either B is given by an equation

$$y + \lambda(z_2, \ldots, z_M) = 0,$$

where $\lambda \in T_p^* W$ is some linear form (possibly a zero one); in that case we say that *B is not pulled back from Q*,

(B) or B is given by an equation

$$\lambda(z_2, \ldots, z_M) = 0,$$

where $\lambda \in T_p^* W \backslash \{0\}$ is some *non-zero* linear form; in that case we say that B *is pulled back from Q*.

Obviously, B is pulled back from Q when and only when $B \in \sigma^*(E_W^*)$ as a point of the dual projective space $B \in E_V^*$. We say that the variety V is *regular* at the point o, where $p = \sigma(o) \in W$, if the following three conditions (R2.1)-(R2.3) hold. Let us fix the system of coordinates $z_1, \ldots, z_{M+1}$ on $\mathbb{P}$ with the origin at the point p, considered above. Restrictions of the polynomials w_i, q_j onto the plane $\{z_1 = z_{M+1} = 0\}$ we denote by the symbols $\bar{w}_i, \bar{q}_j$.

(R2.1) For any linear form $\lambda(z_2, \ldots, z_M)$ the sequence

$$\lambda^2(z_*) - \bar{w}_2, \bar{q}_2, \ldots, \bar{q}_m \tag{4}$$

is regular in $\mathcal{O}_{0,\mathbb{C}^{M-1}}$, that is the system of equations

$$\lambda^2(z_*) - \bar{w}_2 = \bar{q}_2 = \ldots = \bar{q}_m = 0$$

defines a cone of codimension m in $\mathbb{C}^{M-1}$ with the vertex at the origin, and moreover, for the quadratic forms $\bar{w}_2$ and $\bar{q}_2$ we have the estimates

$$\mathrm{rk}\,\bar{w}_2 \geq 4 \text{ and } \mathrm{rk}\,\bar{q}_2 \geq 3.$$

(R2.2) The linear span of any irreducible component of the closed algebraic set

$$\bar{q}_2 = \bar{q}_3 = 0$$

in $\mathbb{C}^{M-1}_{(z_2,\ldots,z_M)}$ is the whole space $\mathbb{C}^{M-1}$.

(R2.3) For $m = 3$ the σ-preimage of any irreducible component of the closed algebraic set

$$\overline{\{\lambda(z_*) = z_1 = z_{M+1} = q_2 = q_3 = 0\}} \subset Q$$

is irreducible for any linear form $\lambda(z_2, \dots, z_M)$. For $m \geq 4$ for any linear form $\lambda(z_*)$ the closed set

$$\sigma^{-1}\overline{(\{\lambda(z_*) = z_1 = z_{M+1} = q_2 = 0\}} \cap Q) \subset V$$

- either is irreducible and reduced,
- or breaks into two irreducible components $B_1 + B_2$, where $B_i = \sigma^{-1}(Q \cap S_i)$ is the σ-preimage of a section of Q by a plane $S_i \subset \{\lambda(z_*) = z_1 = z_{M+1} = 0\}$ and, moreover, $\mathrm{mult}_o B_i = 3$,
- or is non-reduced and is of the form $2B$, where $B = \sigma^{-1}(Q \cap S)$ is the σ-preimage of a section of Q by a plane S and, moreover, $\mathrm{mult}_o B = 3$.

1.3. Correctness of the regularity conditions. It is obvious from the explicit formulation of the regularity conditions, given above, that the property of the variety V to be regular at a point o depends on the pair of polynomials $(f, g) \in \mathcal{F}$ and the point $p = \sigma(o) \in \mathbb{P}$. Let $\mathcal{F}_{\mathrm{reg}}(p) \subset \mathcal{F}$ be the set, consisting of pairs (f, g) for which $f(p) = 0$ and the double cover $V = V(f, g)$ is regular at any (if there are two of them) point $o \in \sigma^{-1}(p)$. If $g(p) \neq 0$, then the regularity is understood in the sense of Sec. 1.1, whereas if $g(p) = 0$, then in the sense of Sec. 1.2. Let $\mathcal{F}_{\mathrm{reg}} \subset \mathcal{F}$ be the set of pairs (f, g), for which the double cover $V = V(f, g)$ is regular at every point $o \in V$.

Proposition 1.1. *The set $\mathcal{F}_{\mathrm{reg}}$ contains a non-empty Zariski open subset in $\mathcal{F}$.*

Proof is given in §4. As usual (see the survey [2]), the idea is to estimate the codimension of the closed set $\mathcal{F}_{\mathrm{non-reg}}(p)$ of double covers, non-regular at the point p. As soon as it is shown that $\mathrm{codim}_{\mathcal{F}} \mathcal{F}_{\mathrm{non-reg}}(p) \geq M + 2$, one can conclude that

$$\overline{\mathcal{F} \backslash \mathcal{F}_{\mathrm{reg}}} = \overline{\bigcup_{p \in \mathbb{P}} \mathcal{F}_{\mathrm{non-reg}}(p)}$$

is a proper closed subset of positive codimension in $\mathcal{F}$, which immediately implies Proposition 1.1.

2 Proof of inequality (2) for a point on the ramification divisor

In this section the inequality (2) is proved in the assumption that the point $o \in V$ lies on the ramification divisor, that is, $p = \sigma(o) \in W$. We use the notations of Sec. 1.2. Proof is obtained by two different methods, depending on whether the hyperplane B is pulled back from Q or not.

2.1. The hyperplane B is not pulled back from Q. In order to prove the inequality (2), assume the converse: the inequality (1) holds, where $D \in |-nK_V|$ is an irreducible divisor, $D^+ \subset V^+$ its strict transform. Let Λ be a pencil of hyperplanes in $\mathbb{P}$, generated by the tangent hyperplanes T_pQ and T_pW^* (in the coordinate form it is the pencil $\alpha z_1+\beta z_{M+1}=0$). Let $T\in\Lambda$ be a general hyperplane.

Set $Q_T=Q\cap T$, $W_T=W\cap T$ and $V_T=\sigma^{-1}(Q_T)$. Obviously, the hypersurface Q_T is non-singular at the point p, the divisor $W_T\subset Q_T$ has an isolated singularity at the point p, so that $o\in V_T$ is an isolated singular point, either, $\sigma_T=\sigma\,|_{V_T}\colon V_T\to Q_T$ is the double cover, branched over W_T. With respect to the affine coordinates $(y,z_2,\dots,z_{M+1})$ the variety V_T is given by the pair of equations

$$y^2-w_2\,|_{\{z_1=-\frac{\beta}{\alpha}z_{M+1}\}}+\ldots=z_{M+1}+q_2\,|_{\{z_1=-\frac{\beta}{\alpha}z_{M+1}\}}+\ldots,$$

where the dots stand for the terms of order 3 and higher in $z_2,\dots,z_{M+1}$. By the regularity condition (R2.1) the quadric $w_2\,|_{\{z_1=z_{M+1}=0\}}$ has rank at least two, so that $o\in V_T$ is an isolated quadratic singularity, where the quadric $E_T=V_T^+\cap E_V$, where $V_T^+\subset V^+$ is the strict transform of the divisor V_T, is given by the equation

$$y^2-w_2\,|_{\{z_1=z_{M+1}=0\}}=0.$$

The latter quadratic form is of rank at least three, so that E_T does not contain hyperplanes in E_V. Therefore, $\mathrm{mult}_o\,V_T+\mathrm{mult}_B\,V_T^+=2$. Since $V_T\in|-K_V|$ is an anticanonical divisor, we get $D\neq V_T$. Thus the effective cycle $D_T=(D\circ V_T)$ of codimension two is well defined, it is an effective divisor on V_T. Obviously, $\mathrm{mult}_o\,D_T\geq 2\,\mathrm{mult}_o\,D$, and the strict transform $D_T^+\subset V^+$ contains the hyperplane section $B_T=B\cap V_T^+$ of the quadric. The quadric B_T is given on E_T by the equation $y+\lambda(z_2,\dots,z_M)=0$. By the regularity condition (R2.1), the quadric B_T is irreducible and reduced.

Lemma 2.1. *The following inequality holds:*

$$\mathrm{mult}_o\,D_T+2\,\mathrm{mult}_{B_T}\,D_T^+>4n. \tag{5}$$

Proof. One has to prove this inequality, because the tangent cone of the divisor D, that is, the algebraic cycle $(D^+\circ E_V)$, can contain the quadric E_T. By the standard formulas of the intersection theory [15],

$$\mathrm{mult}_o\,D_T=2\,\mathrm{mult}_o\,D+2a, \tag{6}$$

where the integer $a\in\mathbb{Z}_+$ is defined by the relation

$$(D^+\circ V_T^+)=aE_T+Z,$$

the effective cycle Z does not contain E_T. On the other hand, every irreducible component of the cycle Z is not contained in the exceptional divisor E_V, that is, $Z=(D\circ V_T)^+$ and thus

$$\mathrm{mult}_{B_T}\,D_T^+ + a\geq \mathrm{mult}_B\,D^+. \tag{7}$$

Combining the estimates (6) and (7), we get (5). Q.E.D. for the lemma.

Now let us consider the standard hypertangent linear systems

$$\Lambda_1^Q,\dots,\Lambda_{m-1}^Q$$

on the hypersurface Q at the point p. In terms of the affine coordinates z_* these systems take the form

$$\Lambda_i^Q=\left|\sum_{j=1}^{i}s_{i-j}(q_1+q_2+\dots+q_j)=0\right|,$$

where $s_a(z_*)$ are arbitrary homogeneous polynomials of degree $a\in\mathbb{Z}_+$ in z_*, $f_j=q_1+\dots+q_j$ is the left segment of length j of the equation f. Set $\Lambda_i=\sigma^*\Lambda_i^Q$ to be the pull back of these systems on V, Λ_i^+ their strict transform on V^+. Obviously,

$$\Lambda_i\subset|-iK_V|,\quad \operatorname{mult}_o\Lambda_i\geq i+1,$$

and by the regularity condition (R2.1) we can say more precisely that $\operatorname{mult}_o\Lambda_i=i+1$, that is, Λ_i^+ is a subsystem of the complete linear system $|-i\varphi_V^*K_V-(i+1)E_V|$. Set also $\Lambda_i^E=\Lambda_i^+|_{E_V}$. These are linear systems of hypersurfaces of degree $(i+1)$ on $E_V\cong\mathbb{P}^{M-1}$.

The regularity condition (R2.1) implies that the base sets of the linear systems Λ_i and Λ_i^E are of codimension i with respect to V and E_V, respectively, and moreover, the codimension does not change when we restrict on V_T and E_T, respectively. These base sets are given by the system of equations

$$q_2|_{\{z_1=z_{M+1}=0\}}=\dots=q_{i+1}|_{\{z_1=z_{M+1}=0\}}=0.$$

Let $(R_2,\dots,R_{m-1})\in\Lambda_2\times\dots\times\Lambda_{m-1}$ be a generic set of divisors of the hypertangent systems. Their strict transforms on V^+ and restrictions onto E_V are denoted by the symbols R_i^+ and $R_i^E=(R_i^+\circ E_V)$, respectively. By the regularity condition the set-theoretic intersections

$$\operatorname{Supp}D_T\cap R_2\cap\dots\cap R_i$$

and

$$\operatorname{Supp}D_T^+\cap E_V\cap R_2^E\cap\dots\cap R_i^E$$

are on codimension i in V_T and E_T, respectively, $i=2,\dots,m-1$. Therefore, the effective cycles

$$Y_{i+1}=(D_T\circ R_2\circ\dots\circ R_i)$$

and

$$Y_{i+1}^E=(D_T^+\circ E_V\circ R_2^E\circ\dots\circ R_i^E)$$

are well defined, and moreover, $Y_i^E=(Y_i^+\circ E_V)$, that is, Y_i^E is the projectivized tangent cone to Y_i at the point o. In particular,

$$\operatorname{mult}_oY_i=\deg Y_i^E=\frac{i!}{2}\operatorname{mult}_oD_T,$$

where the degree of the cycle Y_i^E is understood in the sense of the projective space $E_V \cong \mathbb{P}^{M-1}$.

2.2. End of the proof. Let us use the method of [5] (in a modified form).

Definition 2.1. An irreducible component Z of the cycle Y_i is a B-*component*, if at least one irreducible component of its projectivized tangent cone at the point o, that is, of the algebraic cycle $(Z^+ \circ E_V)$, where Z^+ is the strict transform of Z on V^+, is contained in the hyperplane B.

For the effective cycle Y_m we have the decomposition

$$Y_m = Y_\sharp + Y_B,$$

where in Y_B we collect all B-components of the cycle Y_m, and only them. The following fact is crucial.

Lemma 2.2. *None of the irreducible components of the cycle Y_B is contained in the inverse image of the tangent hyperplane $\sigma^{-1}(T_pQ \cap Q) = \{z_{M+1} = 0\}$.*

Proof. Assume the converse: such a B-component Z does exist, $\sigma(Z) \subset T_pQ$. Then its tangent cone $Z^E = (Z^+ \circ E_V)$ is entirely contained in the tangent cone of the divisor $\sigma^{-1}(T_pQ \cap Q)$. The latter is given in E_V by the equation

$$q_2|_{\{z_1=z_{M+1}=0\}} = 0.$$

By the definition of a B-component, there is an irreducible component S of the cycle Z^E, which is contained in B. Therefore, the polynomials

$$y + \lambda(z_2, \ldots, z_M), y^2 - \bar{w}_2, \bar{q}_2, \ldots, \bar{q}_m$$

vanish on S. Recall that $S \subset E_V$ and $(y : z_2 : \ldots : z_M)$ are homogeneous coordinates on the projective space E_V. Therefore, on S vanish the polynomials

$$\lambda^2(z_2, \ldots, z_M) - \bar{w}_2, \bar{q}_2, \ldots, \bar{q}_m. \tag{8}$$

These polynomials do not depend on y, so that the set of their common zeros is a set of cones in $E_V \cong \mathbb{P}^{M-1}$ with the vertex at the point $(1 : 0 \ldots : 0)$. On the other hand, the hyperplane B does not contain that point, so that S does not contain it, too. Therefore, m homogeneous polynomials (8) in the variables $z_2, \ldots, z_M$ vanish on the irreducible subvariety $\bar{S} \subset \mathbb{P}^{M-2}$, the projection of S from the point $(1 : 0 : \ldots : 0)$. By what was said above, $\dim \bar{S} = \dim S$, so that $\operatorname{codim} \bar{S} = m-1$, which contradicts the regularity condition (R2.1). Q.E.D. for Lemma 2.2.

Let us come back to the proof of inequality (2). By construction, the algebraic cycle Y_m is of degree

$$\deg Y_m = \deg Y_\sharp + \deg Y_B = n \cdot 2m \cdot (m-1)! = 2nm!$$

and its multiplicity at the point o equals to

$$\operatorname{mult}_o Y_m = \operatorname{mult}_o Y_\sharp + \operatorname{mult}_o Y_B = \frac{m!}{2} \operatorname{mult}_o D_T. \tag{9}$$

For the cycle $Y_\sharp$ we have the universal estimate $\mathrm{mult}_o Y_\sharp \leq \deg Y_\sharp$, which cannot be improved. The situation with the cycle Y_B is much better: by Lemma 2.2, the effective cycle

$$Y^* = (Y_B \circ \sigma^{-1}(T_p Q \cap Q))$$

is well defined, and for that cycle we get

$$\deg Y^* = \deg Y_B, \quad \mathrm{mult}_o Y^* \geq 2\,\mathrm{mult}_o Y_B,$$

so that by the universal inequality $\mathrm{mult}_o Y^* \leq \deg Y^*$ we get the estimate

$$\mathrm{mult}_o Y_B \leq \frac{1}{2} \deg Y_B. \tag{10}$$

Finally, by the construction of the cycle Y_m, we have the inequality

$$\mathrm{mult}_o Y_B = \deg Y_B^E \geq \frac{m!}{2} \deg B_T \,\mathrm{mult}_{B_T} D_T^+ = m!\,\mathrm{mult}_{B_T} D_T^+.$$

Combining this inequality with the estimates (9) and (10), we get finally

$$(2\,\mathrm{mult}_{B_T} D_T^+)m! + (\mathrm{mult}_o D_T)m! \leq$$

$$\leq \deg Y_B + 2\,\mathrm{mult}_o Y_\sharp + 2\,\mathrm{mult}_o Y_B \leq$$

$$\leq \deg Y_B + 2\deg Y_\sharp + \deg Y_B = 2\deg Y_m = 4nm!,$$

whence, reducing by $m!$, we get the inequality, which is opposite to (5). This proves the inequality (2) in the case when the hyperplane $B \subset E_V$ is not pulled back from Q.

2.3. The hyperplane B is pulled back from Q. Let us prove the inequality (2), assuming that the hyperplane $B \subset E_V$ is pulled back from Q, that is, in the coordinates $(y, z_2, \dots, z_M)$ it has an equation $\lambda(z_2, \dots, z_M) = 0$, where $\lambda \not\equiv 0$ is some linear form. Again let us assume the converse: $\mathrm{mult}_o D + \mathrm{mult}_B D^+ > 2n$, where $D \in |-nK_V|$ is an irreducible divisor, $D^+ \subset V^+$ its strict transform. Here we argue following the model of the case when the point $o \in V$ lies outside the ramification divisor.

Let Λ_B be the two-dimensional system of hyperplanes in $\mathbb{P}$, cutting out on the tangent space $T_p W$ the hyperplane B. In the coordinates $(z_1, \dots, z_{M+1})$ the equations of these hyperplanes are of the form

$$\alpha z_1 + \beta z_{M+1} + \lambda(z_2, \dots, z_M) = 0,$$

$\alpha, \beta \in \mathbb{C}$ are constants. Let $\Lambda = \sigma^*(\Lambda_B\,|\,_Q)$ be the corresponding linear system on V, $R \in \Lambda$ a general divisor, $Q_R = \sigma(R)$ a smooth hypersurface of degree m in $\mathbb{P}^M$. The branch divisor $W_R = Q_R \cap W$ and the variety R itself are smooth at the points p and o, respectively.

Lemma 2.3. *For the divisor $D_R = D \cap R$ the estimate* $\mathrm{mult}_o D_R > 2n$ *holds.*

Proof. By the formulas of the elementary intersection theory [15] we get: $\mathrm{mult}_o D_R = \mathrm{mult}_o D + \deg Z$, where the effective divisor Z on E_V is defined by the relation $(D^+ \circ R^+) = D^+_R + Z$. By construction of the divisor R, we get $(R^+ \circ E_V) = B$, so that Z contains B with multiplicity at least $\mathrm{mult}_B D^+$. This proves the lemma.

Consider now the pencil Λ_R of hyperplane sections of Q_R, tangent to the branch divisor W_R at the point p. In the coordinate form it is the pencil of hyperplanes $\alpha^* z_1 + \beta^* z_{M+1} = 0$, restricted onto Q_R. Let $T \in \Lambda_R$ be a general divisor of that pencil. Set $W_T = W \cap T$ and $V_T = \sigma^{-1}(T)$. The divisor T is non-singular at the point p, whereas the divisor W_T has at that point an isolated quadratic singularity.

More precisely, the functions $z_2, \dots, z_M$ form a system of local coordinates on Q_R, the tangent hyperplane to the branch divisor W_R is given by the equation $\lambda(z_2, \dots, z_M) = 0$, so that the linear component of the local equation of the divisor T in these coordinates is $\lambda(z_2, \dots, z_M)$. The tangent cone to the divisor V_T in the coordinates $(y : z_2 : \dots : z_M)$ on E_V is given by the pair of equations

$$y^2 - w_2|_{\{z_1 = z_{M+1} = 0\}} = \lambda(z_*) = 0.$$

Therefore, $\mathrm{mult}_o V_T = 2$. The divisor V_T is irreducible and for this reason $D_R \neq V_T$ by Lemma 2.3. Therefore, the effective cycle

$$D_T = (D_R \circ V_T),$$

satisfying the estimate $\mathrm{mult}_o D_T > 4n$, is well defined. The effective divisor D_T on the singular double cover V_T can be assumed to be irreducible.

Lemma 2.4. *The divisor*

$$S = \sigma^{-1}(T_p Q \cap T) = \sigma^{-1}(T_p T)$$

on the variety V_T is irreducible and has multiplicity precisely 4 at the point o.

Proof. By the regularity condition (R2.3) the system of three equations

$$y^2 - w_2|_{\{z_1 = z_{M+1} = 0\}} = \lambda(z_*) = q_2|_{\{z_1 = z_{M+1} = 0\}} = 0$$

defines on $E_V \cong \mathbb{P}^{M-1}_{(y:z_2:\dots:z_M)}$ an effective cycle of codimension 3 and degree 4. Q.E.D. for the lemma.

From the lemma that we have just proven, it follows that $D_T \not\subset \sigma^{-1}(T_p Q)$, or, equivalently, $D_T \neq S$. Consider the second hypertangent system

$$\Lambda_2 = |s_0 f_2 + s_1 f_1|,$$

where recall that $f_1 = q_1$, $f_2 = q_1 + q_2$, $s_i(z_*)$ are homogeneous polynomials in the variables $z_1, \dots, z_{M+1}$. By the regularity condition (R2.3), the base set of its restriction $\Lambda_2^T = \Lambda_2|_T$ has codimension two. Thus for a general divisor $L \in \Lambda_2^T$ we get $\sigma(D_T) \not\subset L$, so that the effective cycle of codimension two on V_T

$$D_L = (D_T \circ \sigma^{-1}(L))$$

is well defined and satisfies the estimate $\mathrm{mult}_o D_L > 12n$. The degree of the cycle D_L (in the sense of the anticanonical class $(-K_V)$) is $2n \deg V = 4nm$. Thus

$$\frac{\mathrm{mult}_o}{\deg} D_L > \frac{6}{\deg V}. \tag{11}$$

However, it follows from the regularity condition (R2.3), that the base set of the linear system $\sigma^*\Lambda_2^T$ either is irreducible and satisfies the equality

$$\frac{\mathrm{mult}_o}{\deg} \mathrm{Bs}\, \sigma^*\Lambda_2^T = \frac{6}{\deg V},$$

or breaks into two components with the same ratio $\mathrm{mult}_o / \deg$. Replacing the cycle D_L by its suitable irreducible component, we may assume that D_L is irreducible and by the inequality (11) $D_L \not\subset \mathrm{Bs}\, \sigma^*\Lambda_2^T$. By the construction of the linear system Λ_2, some polynomial of the form $u_0 f_2 + u_1 f_1$ vanishes on $\sigma(D_L)$, where $u_0 \neq 0$ is a constant, $u_1(z_*)$ is a linear form. Without loss of generality assume that $u_0 = 1$. It follows that

$$f_2|_{\sigma(D)} \equiv -u_1 q_1|_{\sigma(D)}. \tag{12}$$

Lemma 2.5. *The image $\sigma(D_L)$ is not contained in T_pQ.*

Proof. The claim of the lemma means that $q_1 = f_1 = z_{M+1}$ does not vanish on $\sigma(D_L)$. Assume the converse. Then (12) implies that $\sigma(D_L) \subset \mathrm{Bs}\, \Lambda_2^T$, but we already know that this is not the case. Q.E.D. for the lemma.

Consider the effective cycle

$$D^\sharp = (D_L \circ \sigma^{-1}(T_pQ)).$$

It is of codimension 4 on V and satisfies the inequality

$$\frac{\mathrm{mult}_o}{\deg} D^\sharp > \frac{12}{\deg V}.$$

We may assume that the cycle $D^\sharp$ is an irreducible subvariety in V. Its image $\Delta = \sigma(D^\sharp)$ is a subvariety of codimension 4 on Q, satisfying the estimate

$$\frac{\mathrm{mult}_p}{\deg} \Delta > \frac{6}{\deg Q}.$$

The hypersurface Q satisfies the regularity condition (R2.1), so that, repeating the arguments of [16, Sec. 4] word for word (see also [2]), we get a contradiction. The proof of the inequality (2) in the case when $p \in W$ and the hyperplane $B \subset E_V$ is pulled back from Q, is complete.

Note that the last contradiction, completing the proof, can be obtained directly on V, without pushing $D^\sharp$ down on Q, but, on the contrary, pulling back the hypertangent systems

$$\Lambda_i^Q = \left| \sum_{j=1}^{i} s_{i-j} f_j = 0 \right|$$

on V, as it was done above for the second hypertangent system.

3 Proof of the inequality (2) for a point outside the ramification divisor

We use the notations and conventions of Sec. 1.1. Let us prove the inequality (2), assuming that the point $o \in V$ lies outside the ramification divisor of the morphism σ, that is, $p = \sigma(o) \in Q$ lies outside the divisor W. In that case we argue precisely following the scheme of the Fano hypersurfaces [1, §2, Sec. 2.1]. Since our constructions are almost word for word the same as those in [1], we give only the principal steps of the proof.

Let $\varphi\colon V^+ \to V$ and $\varphi_Q\colon Q^+ \to Q$ be the blow ups of the points o and p on V and Q, respectively, $E \subset V^+$ and $E_Q \subset Q^+$ the exceptional divisors. The morphism σ extends to a regular map $V^+ \backslash \{o_1\} \to Q^+$, identifying E and E_Q, where $\sigma^{-1}(p) = \{o, o_1\}$. This identification $E \cong E_Q \cong \mathbb{P}^{M-1}$ will be meant in the sequel without special reservations. Assume that the inequality $\mathrm{mult}_o D + \mathrm{mult}_B D^+ > 2n$ holds, where $B \subset E$ is a hyperplane, $D \in |-nK_V|$, $D^+ \subset V^+$ is the strict transform. The divisor D is assumed to be irreducible and reduced. Let $B_Q \subset E_Q$ be the corresponding hyperplane in E_Q. The exceptional divisor E_Q identifies naturally with the projectivization $\mathbb{P}(T_pQ)$. Let Λ_B be the pencil of hyperplanes in $\mathbb{P}$, cutting out B on E_Q and $\Lambda = \sigma^*(\Lambda_B|_Q)$ the pull back on V of its restriction onto Q. Consider a general divisor $R \in \Lambda$, let $R^+ \subset V^+$ be its strict transform. The divisor R is smooth at the point o, and moreover,

$$R^+ \cap E = B.$$

Set $D_R = D|_R = (D \circ R)$. This is an effective divisor on the variety R.

Lemma 3.1. *The following inequality holds:*

$$\mathrm{mult}_o D_R > 2n. \tag{13}$$

Proof is word for word the same as the proof of Lemma 3 in [1, §2, Sec. 2.1].

Lemma 3.2. *The divisor $T_R = \sigma^{-1}(T_p\sigma(R)) \cap R$ on the variety R is irreducible and has multiplicity precisely 2 at the point o.*

Proof is word for word the same as the proof of Lemma 4 in [1, §2, Sec. 2.1], based on the regularity condition (R1.2).

By Lemmas 3.1 and 3.2, we may assume the divisor D_R to be irreducible and reduced, and different from T_R. Consider the second hypertangent system on Q:

$$\Lambda_2^Q = |s_0 f_2 + s_1 f_1 = 0|_Q,$$

$s_i(z_*)$ are homogeneous polynomials of degree i. The base set of its restriction $\Lambda_2^R = \sigma^* \Lambda_2^Q|_R$ onto R, that is,

$$S_R = \{\sigma^* q_1|_R = \sigma^* q_2|_R = 0\},$$

has by the regularity condition (R1.3) codimension 2 in R and either is irreducible and of multiplicity 6 at the point o, or breaks into two σ-preimages of plane sections

of the hypersurface $\sigma(R)$, each of multiplicity 3 at the point o. In any case, for a general divisor $L\in\Lambda^R_2$ we get $D_R\not\subset L$, so that the effective cycle of codimension two $D_L=(D_R\circ L)$ is well defined and satisfies the estimate

$$\frac{\mathrm{mult}_o}{\deg}D_L>\frac{3}{\deg V}=\frac{3}{2m}. \tag{14}$$

Replacing the cycle D_L by its suitable irreducible component, one may assume it to be an irreducible subvariety of codimension 2 in R, and comparing the estimate (14) with the description of the set S_R given above, we see that $D_L\not\subset S_R$. Now the arguments of Sec. 2.1 in [1] show that this implies that $D_L\not\subset T_R$ (similar arguments are used in the present paper in Sec. 2.3, see the proof of Lemma 2.5). It follows that the effective cycle

$$Y=(D_L\circ T_R)$$

of codimension 4 on V is well defined and satisfies the estimate

$$\frac{\mathrm{mult}_o}{\deg}Y>\frac{6}{\deg V}=\frac{3}{m}. \tag{15}$$

Now a contradiction is achieved by the method of the paper [3]. By linearity of the multiplicity and degree, the cycle Y can be assumed to be an irreducible variety. In the notations of Sec. 1.1 let

$$\Lambda_k=\left|\sum_{i=1}^{K}s_{k-i}f_i+\sum_{i=l}^{k}s^*_{k-i}(y-[\sqrt{g}])_i\right|, \tag{16}$$

$k=1,2,\dots$, be the k-th hypertangent system, where s_j,s^*_j are homogeneous polynomials in z_* of degree j, for simplicity of notations we omit the symbol σ^* (strictly speaking, we should have written $\Sigma\sigma^*s_{k-i}\sigma^*f_i$ etc.) and, finally, the right-hand sum in (16) is assumed to be equal to zero, if $k<l$. The system Λ_k is a subsystem of the complete system $|-kK_V|$ and it is easy to see that $\mathrm{mult}_o\Lambda_k\geq k+1$.

Set

$$\mathcal{M}=[1,m-1]\cap\mathbb{Z}_+=\{1,\dots,m-1\}$$

and $\mathcal{L}=[l,2l-2]\cap\mathbb{Z}_+=\{l,\dots,2l-2\}$. By the regularity condition, for the codimension of the base set of the hypertangent system Λ_k we get

$$\mathrm{codim}\,\mathrm{Bs}\,\Lambda_k=\sharp[1,k]\cap\mathcal{L}+\sharp[1,k]\cap\mathcal{M}.$$

Let

$$(D_1,\dots,D_{m-1},D^*_l,\dots,D^*_{2l-2})\in\prod_{k\in\mathcal{M}}\Lambda_k\times\prod_{k\in\mathcal{L}}\Lambda_k$$

be a general set of hypertangent divisors. Re-order this set as $(L_1,\dots,L_{M-1})$ in such a way that for all divisors $L_i\in\Lambda_{k(i)}$ the inequality $k(i+1)\geq k(i)$ holds. Now the regularity condition implies that

$$\mathrm{codim}_o\,Y\cap L_5\cap\ldots\cap L_k=k,$$

where codim_o means the codimension in a neighborhood of the point o with respect to V. Therefore, one can realize the standard procedure of constructing irreducible subvarieties $Y = Y_4, Y_5, \dots, Y_{M-1}$ of codimension $\mathrm{codim}\, Y_i = i$, where Y_{i+1} is an irreducible component of the effective cycle $(Y_i \circ L_{i+1})$ with the maximal value of the ratio $\mathrm{mult}_o / \deg$. The subvariety $Y^* = Y_{M-1}$ is an irreducible curve on V, satisfying the inequality

$$\frac{\mathrm{mult}_o}{\deg} Y^* \geq \frac{\mathrm{mult}_o}{\deg} Y \cdot \left(\prod_{i=5}^{M-1} \frac{k(i)+1}{k(i)} \right). \tag{17}$$

It is not hard to check that the value of the product in the brackets is

$$\frac{m}{5} \cdot \frac{2l-1}{l} = \frac{m(2l-1)}{5l} > \frac{m}{3}$$

for $l \geq 4$, $3m/8$ for $l = 3$ and $m/3$ for $l = 2$. In any case this value is not less than $m/3$. Since the left-hand side of the inequality (17) is not higher than one, we obtain for the ratio $(\mathrm{mult_o} / \deg) Y$ the estimate, which is opposite to the inequality (15). This contradiction completes the proof of inequality (2).

4 Proof of the regularity conditions

In this section, we prove Proposition 1.1.

4.1. The regularity conditions outside the branch divisor. The condition (R1.1) for a generic cover V was shown in [3]. To prove the condition (R1.2) for a generic hypersurface Q, one needs to argue word for word in the same way as for this condition for a generic Fano hypersurface, see [1,§2], because in this condition only the three components q_1, q_2, q_3 take part. However, the condition (R1.3) for a double cover is essentially different from the corresponding condition for a hypersurface and for this reason needs a special consideration.

Assume at first that $m = 3$. Consider the following general situation: $X \subset \mathbb{P}^N$ is an irreducible subvariety, $x \in X$ a smooth point, $\dim X = k \geq 2$.

Lemma 4.1. *The closed subset $\Xi(x) \subset \mathcal{P}_{2l}(x)$ in the space of homogeneous polynomials of degree $2l$ in the homogeneous variables on $\mathbb{P}^N$, vanishing at the point x, defined by the condition*

$$g \in \Xi(x) \Leftrightarrow g = h^2 \textit{ in } \mathcal{O}_{x,X}$$

is of codimension at least $\binom{2l+k-1}{k-1}$.

Proof. Let $(u_1, \dots, u_N)$ be some system of affine coordinates on $\mathbb{P}^N$ with the origin at the point x, whereas $(u_1, \dots, u_k)$ make a system of local parameters on X at that point. Consider the standard projection

$$\pi \colon \mathcal{O}_{x,X} \to \mathcal{O}_{x,X} / \mathcal{M}_x^{2l+1} \cong \mathbb{C}[u_1, \dots, u_k] / (u_1, \dots, u_k)^{2l+1}, \tag{18}$$

where $\mathcal{M}_x = (u_1, \ldots, u_k) \subset \mathcal{O}_{x,X}$ is the maximal ideal of the local ring. We denote the latter algebra in (18) by the symbol $\mathcal{A}_{2l}$, and its maximal ideal by the symbol $\mathcal{M}$. Let $\Xi \subset \mathcal{M}$ be the set of full squares,

$$\Xi = \{g \in \mathcal{M} |\, g = h^2 \text{ for some } h \in \mathcal{M}\}.$$

Obviously, $\Xi(x) \subset \pi^{-1}(\Xi)$. Furthermore, since the restriction of π onto $\mathcal{P}_{2l}(x)$ is a linear surjective map, we get the estimate

$$\operatorname{codim}_{\mathcal{P}_{2l}(x)} \Xi(x) \geq \operatorname{codim}_{\mathcal{M}} \Xi.$$

It is not hard to estimate the latter codimension from below. Let

$$h = h_1 + \ldots + h_{2l} \in \mathcal{M}$$

be an arbitrary element, decomposed into homogeneous components, that is, homogeneous polynomials in $u_1, \ldots, u_k$. For its square we have the presentation

$$h^2 = \sum_{i=2}^{2l} v_i(h_1, \ldots, h_{i-1}),$$

where the homogeneous component v_i of degree i depends on $h_1, \ldots, h_{i-1}$ only. In particular, h^2 does not depend on the last component h_{2l} and for this reason

$$\dim \Xi \leq \dim \mathcal{M}/\mathcal{M}^{2l}.$$

Therefore, $\operatorname{codim}_{\mathcal{M}} \Xi$ is not less than the dimension of the space $\mathcal{M}^{2l}/\mathcal{M}^{2l+1}$, which is equal to $\binom{2l+k-1}{k-1}$. Q.E.D. for the lemma.

Let us come back to the regularity condition (R1.3) for the double cubic ($m = 3$). Let $Y \subset \mathbb{P}$ be an irreducible component of the closed set

$$\overline{\{q_1 = q_2 = q_3 = 0\}} \cap P,$$

where $P \ni p$ is a hyperplane. Obviously, Y is a cone with the vertex at the point p. The hypersurface W^*_{2l}, that cuts out on Q the branch divisor, does not contain the point p. Therefore,

$$W^* \cap Y \not\subset \operatorname{Sing} Y$$

and we can apply Lemma 4.1 to $X = Y$. We obtain that a violation of the regularity condition (R1.3) at the point p imposes on the polynomial g

$$\binom{2l + M - 4}{M - 4} = \binom{3M - 8}{M - 4} \tag{19}$$

independent conditions. Taking into account that the point p and the hyperplane P are arbitrary, we get that a double cubic of general position is regular at every point outside the ramification divisor, provided that a violation of the condition (R1.3) at a fixed point with a fixed hyperplane P imposes on the polynomial g at least $2M$

independent conditions. It is easy to check that for $M \geq 6$ the right-hand side of (19) is higher (much higher) than $2M$. Q.E.D. for the regularity conditions outside the ramification divisor for $m = 3$.

4.2. Proof for the case $m \geq 4$. In that case our work breaks into two parts: we need to check that the set $\overline{\{q_1 = q_2 = 0\}} \cap Q \cap P$ is either irreducible or breaks into components in the correct way (this part of the condition depends of the polynomial f only), and, furthermore, that the inverse image of each component with respect to σ is irreducible (this part of the condition depends on g only). Let us start with the regularity condition on Q.

As we will see from the arguments below, the estimate for the set of non-regular hypersurfaces Q is the stronger, the higher gets m. So it is sufficient to consider the case $m = 4$. Set

$$F_P = \overline{\{q_1 = q_2 = 0\}} \cap P.$$

This is a quadric of dimension $M - 2$ in $\mathbb{P}^{M-1}$, more precisely, a quadratic cone with the vertex at p. If F_P is a cone over a smooth quadric of dimension $M - 3 \geq 3$, then F_P is a factorial variety, and reducibility of the divisor

$$(q_3 + q_4)|_{F_P}$$

imposes on the polynomial $q_3 + q_4$ at least

$$\binom{M+2}{4} - M^2 + 3$$

independent conditions, which is essentially higher than $2M$. (This estimate is obtained for that type of reducibility, which gives the least codimension of the non-regular set, namely, when a section of the quadric F_P by a quartic with the triple point p breaks into a hyperplane section, containing the point p, and a section of F_p by a cubic with the double point p. For other types of reducibility the codimension is much higher.)

Therefore, we may assume that F_P is a quadratic cone over a singular quadric. Assume that F_P is an irreducible quadric. In that case F_P is swept out by planes of dimension $k \geq M/2$. Moreover, one may choose k-planes of general position $L_1, L_2 \subset F_P$ in such a way that their linear span $< L_1, L_2 >$ is a $(k+1)$-plane $L = \mathbb{P}^{k+1}$, that is, $L_1 \cap L_2 = L_{12}$ is a $(k-1)$-plane, containing the point p and the vertex space of the cone F_P, but not coinciding with the latter (that is, L_{12} is a plane of general position, strictly containing the vertex space of the cone F_p). Let us count the independent conditions on the polynomial $(q_3 + q_4)$, which are imposed by requiring that each of the polynomials $(q_3 + q_4)|_{L_i}$ should be reducible. Elementary but tiresome computations give the codimension

$$\alpha_k = \frac{(k+5)(k+3)k(k-2)}{24} + 1$$

for the restriction of $q_3 + q_4$ onto $L_i \cong \mathbb{P}^k$ to be reducible (again, we mean the type of reducibility which gives the least codimension, namely, into a hyperplane in $\mathbb{P}^k$,

containing the point p, and a cubic with the double point p). Its follows from here that the required codimension for both polynomials $(q_3+q_4)|_{L_i}$ to be reducible, $i=1,2$, is $2\alpha_k-\alpha_{k-1}$, and it is not hard to check that this integer is $\geq 2M$. For $M\geq 7$ it is sufficient to estimate the reducibility of the polynomial $(q_3+q_4)|_{L_1}$, restricted onto one k-plane: this already gives the required codimension.

Finally, assume that the quadric F_P is reducible, $F_P=L_1\cup L_2$, where $L_i\cong\mathbb{P}^{M-2}$ are $(M-2)$-planes. Reducibility of the polynomial q_3+q_4, restricted onto one of the planes L_i, gives, in the previous notations, α_{M-2} independent conditions on the polynomial q_3+q_4, and this number is (considerably) higher than $2M$. This completes the proof of the Q-part of the condition (R1.3).

Let us check the W-part of the condition (R1.3). Let Y be an irreducible component of the closed set $\overline{\{q_1=q_2=0\}}\cap Q\cap P$. We have to estimate the number of independent conditions that follow from reducibility of the double cover $\sigma^{-1}(Y)\to Y$. There are two possible cases: $W\cap Y\not\subset \operatorname{Sing} Y$ (the case of general position) and $W\cap Y\subset\operatorname{Sing} Y$. Let us treat them separately.

If the case of general position takes place, then applying Lemma 4.1, we get at least

$$\binom{2l+M-4}{M-4}\geq\binom{M}{4}>2M$$

independent conditions on the polynomial g, since $\dim Y=M-3$ and $l\geq 2$. If the second case takes place, then the hypersurface W^*_{2l} contains entirely at least one irreducible component S of the set of singular points $\operatorname{Sing} Y$. In that case the simplest thing to do is to apply the method of estimating the codimension from the paper [16]: let $\pi\colon S\to\mathbb{P}^k$, $k=M-4=\dim S$, be a generic projection, then the pull back on S of every non-trivial polynomial on $\mathbb{P}^k$ does not vanish identically. Therefore, the required codimension is not less than the dimension of the space of homogeneous polynomials of degree $2l$ on $\mathbb{P}^k$, that is, again $\binom{2l+k}{k}\geq\binom{M}{4}>2M$. This completes the proof of the W-part of the condition (R1.3).

Therefore, a generic double cover V is regular at every point outside the ramification divisor.

4.3. The regularity condition on the ramification divisor. The fact that a generic double cover V satisfies the conditions (R2.2) and (R2.3), is proven word for word in the same way as the conditions (R1.2) and (R1.3) were proven in the case when the point o does not lie on the ramification divisor. The condition (R2.1) is a somewhat stronger version of the standard regularity condition for the hypersurface $Q\cap\{z_{M+1}=0\}$ at the point p: it involves an additional $(M-1)$-dimensional family of quadratic forms $\lambda^2(z_*)-\bar{w}_2$. Let us consider it in details.

Since the point p lies on the branch divisor, to prove the regularity condition (R2.1) it is sufficient to show that violation of that condition imposes on the pair of polynomials (f,g) not less than $2M-1$ conditions. For the estimates on the rank of the quadratic forms $\bar{w}_2$ and $\bar{q}_2$ this is easy to check. To prove the first part of the condition (R2.1), that is, the regularity of the sequence (4), let us use the following result [17].

Let $u_1,\dots,u_{N+1}$ be independent variables,

$$\mathcal{P}=\prod_{i=1}^{k+1}\mathcal{P}_{m_i}=\{(p_1,\dots,p_{k+1})\},$$

the space of all $(k+1)$-uples of homogeneous polynomials of degree $m_1,\dots,m_{k+1}$, respectively, $0\leq k\leq N-1$, $\mu=\min\{m_1,\dots,m_{k+1}\}\geq 2$. The set $\mathcal{P}$ is a linear space of dimension

$$\dim\mathcal{P}=\sum_{i=1}^{k+1}\binom{m_i+N}{N}.$$

To every $(k+1)$-uple $(p_*)=(p_1,\dots,p_{k+1})$ we correspond the set of its zeros $Z(p_*)\subset\mathbb{P}^N$. Thus a sequence $p_1,\dots,p_{k+1}$ is regular in $\mathcal{O}_{0,\mathbb{C}^{N+1}}$ if and only if $\operatorname{codim} Z(p_*)=k+1$. Let

$$Y=\{(p_*)\in\mathcal{P}|\operatorname{codim}_{\mathbb{P}^N}Z(p_*)\leq k\}$$

be the set of non-regular sets (p_*). Put $I=\{1,\dots,k+1\}$. Set

$$\mu_j=\min_{S\subset I,\sharp S=j}\left\{\sum_{i\in S}m_i\right\}\geq j\mu.$$

In [17] the following fact was shown.

Proposition 4.1. *The following estimate holds:*

$$\operatorname{codim}_{\mathcal{P}}Y\geq\min_{j\in\{0,\dots,k\}}\{(\mu_{j+1}-j)(N-j)+1\}.$$

Applying Proposition 4.1 to our case $N=M-2$, $k+1=m$, $(m_1,\dots,m_{k+1})=(2,2,\dots,m)$, we obtain the required estimate for the codimension of the set of pairs (f,g) that do not satisfy the condition (R2.1). It is easy to see that this estimate is (considerably) stronger than $2M-1$. This completes the proof of the regularity conditions (R2.1)-(R2.3)

5 Application: pencils of Fano double covers

In this section, as an application we study movable linear systems on Fano fiber spaces $V/\mathbb{P}^1$, the fibers of which are double hypersurfaces. Birational geometry of those varieties was studied in [4-6], where as a main tool the traditional quadratic technique of the method of maximal singularities [2,18] was used, the technique that goes back to the classical paper of V.A.Iskovskikh and Yu.I.Manin [19]. Here we prove again the results of [4-6] by the linear method.

5.1. Formulation of the problem and the main result. Let $V/\mathbb{P}^1$ be a Fano fiber space, our assumptions about which are as follows:

(i) V is a smooth projective variety with the Picard group $\operatorname{Pic}V=\mathbb{Z}K_V\oplus\mathbb{Z}F$, where F is the class of a fiber of the projection $\pi\colon V\to\mathbb{P}^1$,

(ii) every fiber $F_t = \pi^{-1}(t)$, $t \in \mathbb{P}^1$, is a Fano double hypersurface, $F_t \in \mathcal{F}$, and moreover, F_t is regular at every smooth point (in particular, every smooth fiber $F_t \in \mathcal{F}_{\mathrm{reg}}$),

(iii) if $o \in F_t$ is a singular point, then o is an isolated quadratic singularity (in particular, there are finitely many such points), satisfying the corresponding regularity condition (R1.4) or (R2.4), depending on whether the point o lies outside the ramification divisor or on that divisor; the conditions (R1.4) and (R2.4) are formulated below.

Theorem 3. (i) *Assume that $\Sigma \subset |-nK_V + lF|$ is a movable linear system on V and $l \in \mathbb{Z}_+$. Then the following equality holds:*

$$c_{\mathrm{virt}}(\Sigma) = c(\Sigma) = n.$$

(ii) *Assume in addition that the Fano fiber space $V/\mathbb{P}^1$ satisfies the K-condition:*

$$-K_V \notin \mathrm{Int}\, A^1_{\mathrm{mov}} V,$$

where $A^1_{\mathrm{mov}} V \subset A^1_{\mathbb{R}} V \cong \mathbb{R}^2$ is the closed cone generated by the classes of movable divisors on V. Then the fiber space $V/\mathbb{P}^1$ is birationally superrigid, in particular, the projection $\pi: V \to \mathbb{P}^1$ is the only non-trivial structure of a rationally connected fiber space on V.

Proof. It is sufficient to check the first claim: the second one follows from it in a straightforward way. Assume that $c_{\mathrm{virt}}(\Sigma) < c(\Sigma)$, then the linear system Σ has a maximal singularity $E \subset V^+$, where $\varphi: V^+ \to V$ is a birational morphism, E is an exceptional divisor. Its centre $B = \varphi(E) \subset V$ is an irreducible subvariety of codimension 2 or higher. Let $F = F_t$ be some fiber, intersecting B. If $B \cap F$ is not a singular point of the fiber F, then, restricting the system Σ onto the fiber F, we get a contradiction with Theorem 1: for a generic divisor $D \in \Sigma$ the pair $(F, \frac{1}{n}D|_F)$ is not canonical. Therefore, one may assume that $B = o \in F$ is a singular point of a fiber. As we will show below in Sec. 5.2-5.3, this possibility does not realize, either, if the corresponding regularity condition (R1.4) or (R2.4) holds. Therefore, the linear system Σ cannot have maximal singularities, which proves the theorem.

Remark 5.1. As the proof given above shows, the linear method does not make use of the condition of twistedness of the fiber space $V/\mathbb{P}^1$ over the base. Essentially, all the proof is restricting to some fiber and applying the property of divisorial (log) canonicity of every fiber. All work is accumulated in the proof of that property, which requires stronger regularity conditions than birational rigidity. The regularity conditions, on which the quadratic method is based, are weaker and for that reason the results obtained by the quadratic method, are more precise. However, the quadratic constructions require considerably more work.

5.2. Singular points of fibers. Let $o \in F = F_t$ be a singularity of the fiber $\pi^{-1}(t)$. We say that the point o is a singularity *of the first type*, if it lies outside the ramification divisor of the morphism $\sigma: F \to G \subset \mathbb{P}$, that is, $p = \sigma(o) \notin W$. In that case we assume that the hypersurface G in suitable affine coordinates with the

origin at the point p is given by the equation

$$f = q_2 + \ldots + q_m = 0,$$

where $q_2(z_1, \ldots, z_{M+1})$ is a non-degenerate quadratic form. The branch divisor of the morphism σ is cut out on G by the hypersurface

$$g = 1 + w_1 + \ldots + w_{2l} = 0,$$

which does not pass through p. We say that the point o is a singularity *of the second type*, if it lies on the ramification divisor of the morphism σ, that is, $p = \sigma(o) \in W$. In that case G is given by the equation

$$f = q_1 + \ldots + q_m = 0,$$

where $q_1 \equiv z_{M+1}$, and the equation of the branch divisor is of the form

$$g = \alpha z_{M+1} + w_2 + \ldots + w_{2l} = 0,$$

where $\alpha \in \mathbb{C}$ is an arbitrary constant.

Let us consider first the singularities of the first type. We say that the variety F is *regular* at the point o of the first type, if (in the notations above) the following condition holds:

(R1.4) for any non-zero linear form $\lambda(z_1, \ldots, z_{M+1})$ for $m \leq 2l$ the sequence

$$\lambda, q_2, \ldots, q_m, h_{l+1}, \ldots, h_{2l-1}$$

is regular in the local ring $\mathcal{O}_{0,\mathbb{C}^{M+1}}$, and for $m > 2l$ the sequence

$$\lambda, q_2, \ldots, q_{m-1}, h_{l+1}, \ldots, h_{2l}$$

is regular in the ring $\mathcal{O}_{0,\mathbb{C}^{M+1}}$, where $h_j(z_*)$ have the same meaning as in the regularity condition (R1.1) and, moreover, the closed set $\sigma^{-1}(\{\lambda = q_2 = 0\} \cap G)$ is irreducible.

Since there are finitely many singular points, the regularity condition does not require a special proof (it is sufficient to require that the system of equations

$$q_2 = \ldots = q_m = h_{l+1} = \ldots = h_{2l-1} = 0$$

or, respectively, the one with q_m replaced by h_{2l}, determines a closed set, none of the components of which is contained in a hyperplane; the second part of the regularity condition holds at a point of general position in an obvious way).

Let $D \in |-nK_F|$ be a prime divisor, $\beta\colon F^+ \to F$ the blow up of the point o, $\beta_G\colon G^+ \to G$ the blow up of the point p on G, $E_F \subset F^+$ and $E_G \subset G^+$ the exceptional divisors (non-singular quadrics), where E_F identifies naturally with E_G. Let $B \subset E_F$ be a hyperplane section. For the strict transform $D^+ \subset F^+$ we have $D^+ \sim -nK_F - \nu E_F$, where $\nu \in \mathbb{Z}_+$ is some integer.

Proposition 5.1. *The following inequality holds:*

$$\nu + \mathrm{mult}_B D^+ \leq 2n. \tag{20}$$

Proof. Assume the converse. Let $R \subset \mathbb{P}$ be the unique hyperplane, $R \ni p$, cutting out B on $E_G \cong E_F$, say, $R = \{\lambda(z_*) = 0\}$. The divisor $T = \sigma^{-1}(R \cap G)$ satisfies the inequality (20) with $n = 1$ and is irreducible, so that $D \neq T$ and one may form the scheme-theoretic intersection $Y = (D \circ T)$, an effective cycle of codimension 2, satisfying the inequality

$$\frac{\mathrm{mult}_o}{\deg}(D \circ T) > \frac{4}{\deg F} = \frac{2}{m}.$$

By linearity, we may assume that the cycle Y is an irreducible subvariety of codimension 2 on F, or a prime divisor on T. Set D_2 to be the divisor on F, cut out by the hypersurface $q_2 = 0$. By the regularity condition, $(D_2 \circ T)$ is a prime divisor on T, and moreover,

$$\frac{\mathrm{mult}_o}{\deg}(D_2 \circ T) = \frac{3}{2m}.$$

Therefore, $Y \neq (D_2 \circ T)$, so that $Y \not\subset D_2$ and we may form the effective cycle $(Y \circ D_2)$ that has an irreducible component $Y^\sharp$, satisfying the inequality

$$\frac{\mathrm{mult}_o}{\deg} Y^\sharp > \frac{3}{m}.$$

The subvariety $Y^\sharp$ is of codimension two on T. The condition (R1.4) can be understood as the regularity condition for the variety T. Applying the technique of hypertangent divisors on T to the subvariety $Y^\sharp$ in the standard way, we obtain a contradiction (intersecting one by one with hypertangent divisors, we construct a curve $C \subset T$, satisfying the inequality $(\mathrm{mult}_o / \deg) C > 1$, which is impossible). Q.E.D. for Proposition 5.1. The case of a singularity of the first type is excluded.

5.3. Singularities of the second type. We say that the variety F is *regular* at a point o of the second type, if the following condition holds.

(R2.4) For any linear form $\lambda(z_1, \ldots, z_M)$ the sequence

$$\lambda^2(z_*) - \bar{w}_2, \bar{q}_2, \ldots, \bar{q}_m$$

is regular in $\mathcal{O}_{0,\mathbb{C}^{M+1}}$, where $\bar{w}_i, \bar{q}_j$ stand for the restrictions of the polynomials w_i, q_j onto the hyperplane $z_{M+1} = 0$, whereas the quadratic forms $\bar{w}_2$ and $\bar{q}_2$ have the full rank, the closed set $\{\bar{q}_2 = \bar{q}_3 = 0\}$ in $\mathbb{C}^M$ is irreducible and its linear span is the whole space $\mathbb{C}^M$, whereas the closed set

$$\sigma^{-1}(\overline{\{\lambda(z_*) = q_1 = q_2 = 0\}} \cap G)$$

is irreducible and reduced.

Note again, that since there are finitely many singular points, the condition (R2.4) does not need a special proof.

Let $D \in |-nK_F|$ be a prime divisor, $\beta: F^+ \to F$ and $\beta_G: G^+ \to G$ the blow ups of the points o and $p = \sigma(o)$, respectively, with the exceptional divisors $E_F \subset F^+$ and $E_G \subset G^+$. The morphism σ extends to the double cover $\sigma^+: F^+ \to G^+$, where $\sigma_E: E_F \to E_G$ realizes the quadric E_F as the double cover of $E_G \cong \mathbb{P}^{M-1}_{(z_1:\ldots:z_m)}$ branched over $\{\bar{w}_2 = 0\}$. Let $D^+ \subset F^+$ be the strict transform of the divisor D and define the integer $\nu \in \mathbb{Z}_+$ by the equivalence $D^+ \sim -nK_F - \nu E_F$. Let $B \subset E_F$ be a hyperplane section of the smooth quadric E_F. Repeating the arguments of §2 word for word (considering the cases when B is pulled back and not pulled back from G separately), we obtain the claim of Proposition 5.1. This excludes the case of a singularity of the second type. Proof of Theorem 3 is complete.

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